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In recent years there has been an increasing interest in "new math." Accompanying this interest has been a great deal of confusion and perplexity for parents and teachers concerned with determining its worth. Five articles deal with the new programs. A professor of mathematics sees the new mathematics programs as one of the most novel and exciting things going on today in American education. An associate professor of physics reflects the critical viewpoint that the new math has only created a situation of confusion and offers little more than a status symbol for mathematicians to gain grants or publishers to sell books. While the views remain varied, the publishers of this pamphlet feel that this variation reflects the current status of informed thinking on this subject.

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by

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Five Views of the "New Math"

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COUNCIL FOR BASIC EDUCATION
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INTRODUCTORY NOTE

In recent years there has been increasing interest in "new math." Advocates have said that it represents a marked improvement over previous programs. Many people—not only parents but teachers too—have admitted being confused or perplexed. Some laymen and a few experts have expressed concern that it may be leading to a mathematical illiteracy comparable to the reading illiteracy which resulted from the "whole word" or "look-say" approach to beginning reading.

In order to shed some light on the matter, the Council asked Edwin E. Moise, James B. Conant Professor of Education and Mathematics at the Harvard Graduate School of Education, to write a short article. It first appeared in the November 1964 issue of the *CBE Bulletin* and is reprinted in slightly expanded form in the following pages. The Council then submitted Professor Moise's article to four other professors who represent various points of view on the "new math." The resulting articles are also presented herewith.

The reader will notice that the five articles deal with the new programs in various ways and are laudatory or critical in varying degrees. The Council feels that this variation reflects the current status of informed thinking on the subject. There are many "new" programs. Secondary schools, with their more specialized teachers and more structured curricula in mathematics, present a different picture from that of the elementary schools. Implementation of the programs involves problems additional to those of constructing the programs; particularly important here is the education and competence of the classroom teachers involved. Some of the programs have been carried out under carefully controlled circumstances quite different from those prevailing in the typical school. Some of the applications have been supplemental to the previous mathematics programs; others have been substitutes.

In addition to the five articles, this paper contains a list of Suggestions for Further Reading.

Five Views of the "New Math"

Edwin E. Moise

(Dr. Moise is James B. Conant Professor of Education and Mathematics at the Harvard Graduate School of Education. He has served on the tenth-grade geometry writing team of the School Mathematics Study Group. This article first appeared in the November 1964 issue of the CBE Bulletin, *and is reprinted here in slightly expanded form.)*

In the past few years, the teaching of mathematics has rapidly changed. It is natural enough for the public to be confused about this, because there has been some confusion about it within the profession. Partly this is due to the nature of the subject: many otherwise educated people do not understand mathematics, even in a rudimentary way. Partly it is due to misleading slogans, such as the *new mathematics* and *modern mathematics*. To understand what is going on, the first thing that we need to recognize is that while the programs are new and modern, the mathematics contained in them is not. In high schools, there have been some changes in subject-matter, but these are the smallest and least important of the innovations. The most important changes have been in the style in which the old content has been formulated and presented.

It is in the elementary schools that the reforms have been most radical; and this is where radical reforms were most needed. Surely good teaching has been done for a long time, in many places, but in general the elementary teaching of mathematics has been a disaster. The most obvious evidence of this is the image of the mathematician in the popular culture. Normal people, in fiction, never turn out to be mathematicians. Most people think of mathematics as dismal, and they got this impression not from anti-mathematics propaganda, but from their own experience with it, in the schools.

Traditionally, elementary school mathematics has been arithmetic and nothing else; and arithmetic has been a series of techniques of calculation, and nothing else. The new programs introduce much more variety. They present (usually in an informal and intuitive style) various ideas in geometry, algebra, and the theory of numbers.

And the approach to arithmetic itself is quite different. It is being presented not as a set of procedures but as a body of knowledge.

There has been a great deal of publicity about the so-called "theory of sets." The general impression is that this is a new theory, which is being taught instead of arithmetic. In fact, the idea of a set (or collection, or family) of objects is very familiar in common speech, by way of various synonyms: a set of dishes is a set; a set of chessmen is a set; a pack of cards is a set of cards; and so on. In fact, the idea of a set must be older than any known language, living or dead. In all of the Indo-European languages, the words for *one*, *two*, and *three* are obviously cognate. Therefore the numbers that we use for counting were known in the Stone Age. And this means that the idea of a set was part of Stone Age mathematics, because sets are the objects to which the counting numbers are applied. If you look at a pile of three pebbles and a herd of three mastodons, and describe them by the same number *three*, you surely are not describing a common property of pebbles and mastodons, considered one at a time. What you are describing is a common property of two sets; each of the sets has three members. Thus the idea of a set is implicit in first-grade arithmetic. In the new programs, this latent idea is brought to the surface, made explicit, and furnished with a name.

This is typical of the way in which the "new" mathematics looks new but isn't. It is also a good example of the purposes of innovations in language. We are trying to teach ideas; and it is very hard to teach ideas which have no names. The results can be bewildering (and have been) to parents who try to help children with their homework, and find that they can't even recognize the mathematics that they thought they knew. But there is a simple remedy for the language problem: when you are examining a mathematics book, you should read the pages in the order of their appearance, starting with page one. Another trouble, of course, is that the ideas underlying arithmetic are intellectually new, even to most well-educated adults. Surely they can be learned, and they need to be, for most of the present generation of parents didn't learn them in school.

It is hard to make an exact judgment of the results of the new style of teaching. One of the troubles with "educational research" is that the things that can be measured exactly are not the things that matter most. Thus, for example, it is easy to find out whether a student knows that Leonardo da Vinci is dead, but it is hard to

find out in what sense (if any) the student understands Leonardo's paintings. In arithmetic, what is easy to test is manipulative skill. Tests have shown that the elementary textbooks of the School Mathematics Study Group teach the children to calculate just as well as books which do not attempt to teach them anything else. Everything else that happens in the SMSG program must be net profit. But nobody has gotten an exact measure of the latter. One thing was obvious, however, as soon as the books were written, and before they were tried: the improvement in intellectual content was so great that they surely would produce either an educational improvement or a collapse of classroom morale. The latter has not occurred; the new programs in general are far more popular than the old ones.

The above may seem like special pleading, but in fact it is not. I have worked with the SMSG (for three summers, on the tenth-grade geometry writing team), but I am not a press agent, for them or for anybody else. Books and opinions vary; and every textbook that I have read has some features which look bad to me. But in a short general article the most that I can hope to do is to convey some idea of what the new programs mean in principle and in general outline. In this spirit, the new programs can fairly be described as an intellectual renascence.

In saying this, I do not mean to suggest that every elementary school should introduce one of the new programs next year. It seems likely that there are many otherwise well-trained teachers who would do less well than they are doing now if they were asked to teach a new mathematics curriculum without notice, preparation, or help. But in the presence of a will to do the job, a rather small amount of in-service help may be enough, along with the help furnished by the teachers' manuals and the texts themselves. And if even one well-qualified teacher is available for occasional consultation, this may make quite a difference in the morale of a whole staff. There is no simple prescription for striking a balance between enterprise and prudence in carrying out such a job as this: and if we waited for ideal conditions, we would not be doing what we can do.

The significance of the reforms at the high school level is somewhat similar, but is harder to explain, for lack of simple examples. It involves a philosophical re-orientation. A few years ago, when I was talking mathematics with a colleague, his wife heard one of us mention "a theorem in algebra." She was puzzled, and asked what on earth this could mean, pointing out that "there are no theorems

in algebra!" The speaker was intelligent, educated, and in many ways brilliant; and her remark is a fair description of what algebra has looked like in conventional courses. To see its significance, we should remember the meaning of the word *theorem*: a theorem is simply a mathematical statement which is known to be true. My friend's wife was saying that algebra did not contain any statements which were known to be true, and she had every right to draw this conclusion from the course that she had taken in the ninth grade. Algebra has commonly been taught as a collection of procedures and behavior patterns. Early in the course, if the "problem" is $(x - 3)\ (x + 3)$, then the "answer" is $x^2 - 9$; and later, if the "problem" is $x^2 - 9$, then the "answer" is $(x - 3)\ (x + 3)$. In fact, algebra is the study of number systems, in the same way in which geometry is the study of planes and space; it has factual content, just as geometry does; and the manipulative processes of algebra depend on the properties of the number system. In the formula $x^2 - 9 = (x - 3)\ (x + 3)$, x is not a "general number"; in fact there is no such thing as a general number. We use such letters as x, y, and z merely to mark the spots where numerals are to be inserted, and when we write the formula above, we mean that no matter what we substitute for x, we get a true equation. For example,

$$5^2 - 9 = (5 - 3)\ (5 + 3),$$
$$7^2 - 9 = (7 - 3)\ (7 + 3),$$
$$11^2 - 9 = (11 - 3)\ (11 + 3),$$

and so on for infinitely many other cases.

Equations of this kind, involving the use of letters, are called *open sentences*. When we substitute a numeral for x, we get a *sentence*. Thus "$x^2 - 9 = 0$" is an open sentence. In this case the only way to get true sentences is by substituting 3 or -3 for x.

In other words, we use the letter x in algebra as a *typographical* device, for two purposes, namely to make general statements of the form "For *every* x, ... is true," and to ask questions of the form "For *which* x is it true that . . .?"

This may look like a trivial point, but it is not. One of the paradoxes of mathematical education is that the students entering advanced courses do not seem to understand the ideas on which the earlier courses should have been based. The reason is simple: at many stages, the earlier courses abandoned the attempt to teach

the underlying ideas, and tried instead to teach skills in a purely behavioral sense. This was not a matter of pedagogic artistry. Time and again, when teachers are being trained to handle the new materials, it is obvious that many teachers accustomed to the old tradition are learning, for the first time, the intellectual substance of the courses that they had been teaching for years. A mathematician, looking hastily at the old books, may easily read between the lines, and infer that of course they mean thus and so (i.e., whatever they ought to mean). But many of the teachers made no such inference, and neither did their students.

All this is meant to be a sample of the sort of changes that are being made, and a suggestion of the reasons for them. It is surely not an adequate exposition of the substance of the reforms, let alone a "proof" that they are worthwhile. Indeed, if the reforms really are worthwhile, then no short article could be an adequate exposition of them; if a few paragraphs were enough, even for adults, then we would hardly need a whole series of new books, even for children. What I have been trying to describe is a change in the quality of an experience which requires twelve years of study. There is good reason to hope that the new experience will enable students to see the real face of mathematics. If this happens—and I believe that it will—then the future users of mathematics will use it better, and more of our potential mathematicians will find out, in school, that that is what they are. And if real mathematics is taught to everybody, then it may earn again the position which it earned in Plato's time, as a part of liberal education.

Alexander Calandra

(Dr. Calandra is Associate Professor of Physics at Washington University, St. Louis. He is also Executive Director of the American Council for Curricular Evaluation and a frequent critic of the "new math.")

Modern Mathematics. The expression "modern mathematics" was first used for the content of those branches of mathematics developed since the turn of the 19th century. It is in this sense that it often appears in the applications of mathematicians for grants.[1] As the expression "modern mathematics" began to develop unfavor-

[1] This can be seen in the discussions which take place in Congressional hearings relating to appropriations for curricular revision. These are available in the reports entitled "Independent Offices Appropriations" for any given year. The chairman of the House Subcommittee on Independent Offices is the Honorable Albert Thomas of Texas.

able connotations, promoters of reform began to disassociate themselves from it with statements such as the one by Professor Moise: "The most important changes have been in the style in which the old content has been formulated and presented."

Since this transition from an emphasis on content to an emphasis on form takes place at the very time that mathematically unsophisticated educators have been convinced of the advantages of a change in content, considerable confusion has been created. This confusion is intensified by publishers who label all their books modern, new or contemporary. Thus it comes about that the expression "modern math" is often little more than a status symbol used by mathematicians to obtain grants, educators to gain prestige, and publishers to sell books. This combination of influences has given rise to a promotional exuberance which is hardly justified by what is offered. Let us examine three of these offerings.

1. Set Notation. A common denominator of many of the new programs is the use of set notation. At best, the value of this is questionable, and it is hard to see it as more than the enthusiasm of subject-matter specialists for their specialty without too much relevance to the needs of school children. Support for this statement can be found in the writings of leading mathematicians in America, Australia, Canada, England, Germany, Switzerland and other countries.[2] One of the milder statements is that of Professor R. L. Goodstein, formerly editor of the *British Mathematical Gazette*:

> "The reduction of relations to ordered pairs [and then to sets] is a technical device of interest in the formalisation of set theory, but is nonsensical out of its proper setting . . . Proposals as extreme and eccentric as those under review can I fear only serve to damage the case for reform."

Some scholars have ridiculed the pretentiousness of the set approach by lampooning it. Professor W. W. Sawyer of Wesleyan University says:

> "One writer urges that the students 'become active participants in an adventure in the learning of concepts.' And what is this adventure? The students give the teacher their own examples of sets. They begin with sets of rather similar things; but soon the adventure has risen to the pitch where they can contemplate such sets as 'the nose of the notary, the moon, and

[2] More complete statements on this point are available in *Modern Mathematics—Some Critical Appraisals*, 1965, The American Council for Curricular Evaluation, 829 Woodruff Drive, Ballwin, Missouri.

the number 4.' The students have been led with great pedagogical skills to the breath-taking conclusion that any collection of things is a collection."

Professor R. A. Staal of the University of Waterloo (Canada), in a masterpiece of satire, mimics modern elementary mathematics texts with:

"Oh, see. Johnny has a set of marbles. See Johnny's set. Look, look, Billy has a set of marbles. See Billy's set. Here comes Mary. Mary gets all the marbles. Mary gets the union of Johnny's set and Billy's set. See Mary's union."

A strong statement by the mathematics department of Western Reserve University contains the cogent remarks:

"Sophistication in trivial matters and superficial treatment of difficult ideas make a fetish of a facade of logical perfection. . . . Lack of editorial responsibility resulting from anonymous authorship . . ."

2. *Open Sentence.* The idea of an open sentence has some merit as a casual analogy but it loses its didactic value when it is stressed to the point of making it appear that the validity of mathematics is based on the structure of verbal language. Furthermore, the fact that a highly competent leader of the reform movement, Professor Paul Rosenbloom of the University of Minnesota, seems to take a dim view of its merit does not inspire much confidence in it. He has noted:

"In the writing teams for grades 9-11, some of us were disappointed that the people who we had thought would be advocating applications came up with writing stuff about open sentences and the like."

3. *Number and Numeral.* A major defect of many of the modern mathematics programs is the labored pedantry which is used to give the impression of deep mathematical insight. The emphasis on the distinction between number and numeral (the name of a number) is an example of this. The nonsense in this emphasis is obvious when verbal language is used with the same affected precision.

Teacher: Who are you?

Student: I am Robert Jones.

Teacher: Oh, dear. You must never say that. You should say that you are the boy whose name is Robert Jones. After all, you do not mean to say that you are made of letters.

Innovators, administrators and teachers who become ecstatic about this distinction between number and numeral are in much the same position as Moliere's Monsieur Jourdain, who was thrilled to discover that he spoke prose.

Self-evaluation. Evaluations of the modern mathematics movement by the promoters themselves are occasionally quoted to show that this approach does not lead to deficiencies in routine skills. This type of evaluation leaves something to be desired, and is subject to the kind of weakness that was inherent in the evaluation of Krebiozen by the developers of that product. In addition, the Hawthorne effect (an improvement in outcome occurring because of a conscious or unconscious increase in effort or enthusiasm) stemming from the prestige of the promoters makes the ambiguous results obtained open to rather unflattering interpretations.

The most serious criticisms of modern math are inadvertently made by the promoters themselves. One of these is the effort to train the parents to teach what the teachers themselves, in a much more suitable environment, cannot. Another is the pious reference to the new, new mathematics or the second wave of mathematical reform. For the most part these appear to be euphemisms for saying that the original reform was grossly misdirected. Still another is the cavalier admission of educational brinkmanship in Professor Moise's article: "One thing was obvious, however, as soon as the books were written, and before they were tried: the improvement in intellectual content was so great that they surely would produce either an educational improvement or a collapse of classroom morale." Although the candor of this remark is admirable, its implications are frightening.

In fact, it is by no means clear that there has *not* been a collapse of classroom morale in a substantial number of cases. Some candid comments to this effect have appeared in the *New York Times*:

> "Dr. Max Beberman, the noted University of Illinois pioneer in the field . . . told mathematics teachers in Montreal last month that 'a major national scandal' may be in the making because of hasty introduction of the new mathematics in the elementary schools. He said this was harming the teaching of arithmetic to children who would need its skills for the tasks of adult life. A survey by The New York Times of other leaders in mathematics education found this concern widely, though not unanimously, held. . . . Prof. Paul Rosenbloom of the Minnesota Mathematics and Science Teaching Project said: 'I have seen

some very shocking things taking place. Certain kinds of nonsense have been spreading very rapidly.' He assailed some commercial textbook publishers who, he said, have been hiring 'hacks to do the same old rote job' on the new mathematics as was done in texts for the old mathematics."[3]

Another unfelicitous aspect about the statement that "the improvement in intellectual content was so great" is that it would appear to contradict Professor Moise's introductory statement that "the most important changes have been in the style."

Summary Evaluation. The total evaluation of the modern math movement is a difficult undertaking, for it requires statements that apply to the bulk of the more than six hundred programs subsumed under this title. These programs range from some rather wearying pedantic rubbish to some occasionally lively and interesting approaches. If I have unintentionally jeopardized some of the latter in such a short general note I want to apologize here and now, but I would rather do this than to stand by and merely watch the over-all academic mayhem that is going on in the name of reform.

In many ways the new math movement has the character of the Children's Crusade of the Middle Ages. It is recognized as such by many responsible educators, but it is difficult to stop because of the very large and tightly knit web of vested interests preying on the mathematical unsophistication of the press, the public, and the foundations themselves. Under these circumstances, I urge administrators to forget the prestige of the sponsors and view with restrained enthusiasm anything which does not really make sense to them, rather than be Sputnik-panicked into the hysterical adoption of new programs. Administrators might consider opening and closing each curriculum meeting with the following quotations from two famous mathematicians:

"It is a safe general rule to apply that when a mathematician talks with a misty profundity he is talking nonsense."

—A. N. Whitehead

"It is very easy to be impressed by something you do not understand too well."

—G. H. Hardy

[3] The *New York Times*, January 25, 1965.

Robert B. Davis

(Dr. Davis is Director of the Madison Project, which has developed one of the "new math" programs for elementary schools. He is also Professor of Mathematics and Education at Syracuse University and Webster College.)

The popular interest in "new mathematics" is a gratifying response to what is, in fact, one of the most novel and exciting things going on today in American schools and colleges. Yet there is one shortcoming which is worrisome: "new mathematics," as it appears in most discussions, is a shallow affair: the true depth of what is happening is seemingly not recognized.

To look at the matter in some perspective, we might divide schools into three categories.

A very large number, and possibly a majority, of schools have really not been touched by "new mathematics" at all. The same can be said for most college programs to educate future teachers.

Another large segment of schools (and also of textbooks) have been brushed very lightly. The word "discovery" has been added, as have various other slogans (such as "emphasis on structure"), but these words and slogans are not related to practice. A few currently-fashionable topics (such as "sets" and "binary numerals") have been somehow appended to the school program, but they serve little essential functional purpose. Many of the "new" topics are not well-understood, nor is the reason (if any) for including them.

There is a third category: a significant minority of schools are teaching a genuinely modified and improved version of "mathematics." Notations are clearer, definitions more wisely chosen, student participation is greater and is focused more toward honest essentials, the sequence of topics is improved, important and appropriate new topics have been added, relatively unimportant (or even wrong) material has been deleted, the pace is better adjusted to actual student potential, the quality of expository writing has been improved, relations to other subjects are more adequately treated, and—almost most important of all—a more creative and exciting flavor permeates the program. Moreover, in these schools teacher comprehension is adequate to both the mathematical and the pedagogical task.

Actually, it is a mis-statement to say "in these schools." It would be more accurate to say "in these classrooms"—for one

of the persistent and troublesome problems has been the "spotty" character of excellence. It may be achieved in one school in the fourth grade, but not followed up in the fifth. It may be achieved in an elementary school, but not in the junior high school to which the students move next. It may be achieved in grade 12 in one building, but not achieved in grade 9 in that same building—and so on. Problems of articulation involve the task of achieving a cooperative effort—even a cooperative *devotion*—by a sizeable group of people; hence they are always difficult. Articulation problems are vastly magnified and compounded by a faster pace of change, and by a greater insistence on excellence. Hence they are nowadays acute, if not devastating.

However, we have not yet probed the true depth of today's curriculum reform, and this is the main point I wish to make.

When M.I.T.'s Zacharias and Friedman, when Illinois's Page and Beberman, when Nobel prize winners, when leading scientists and equipment designers and film makers and teachers descended upon the pre-college curriculum (supported by far-sighted foundations), they brought with them many ideas, values, prejudices, and beliefs that were often quite at variance with usual pre-college notions. The impact of this confrontation (or collision) can still be felt in tremors and vibrations that portend a *very* different future for American education, if we can understand what is *really* happening, and if we have the wisdom and courage to build upon it and to grow with it.

Some of the new ideas are relatively simple and easily identified.

One relates to the rate of change in education: prior to the impact, a textbook series was several years in preparation, was not changed for 10 or 15 years or longer, and was often adopted for five years at a time. The new practice has often been to prepare books in a few months (at, admittedly, far greater cost), and to print them in paper-back versions designed to be *worn-out* in one year!

Another difference was cost: M.I.T. spent $7,000,000 developing its one-year high school physics course—and, by most real criteria, the result is a bargain. Perhaps never before had so much *change* and so much *quality*, per dollar, been purchased for education. In terms of *innovation* and *excellence*, the M.I.T. physics course has proved to be the large-size economy package.

Another difference was teacher education: many of the outstanding teachers of the "new curriculum" movement had educational backgrounds entirely different from the usual education of teachers. Indeed, if they had not been employed by M.I.T. or Yale or the University of Illinois or Syracuse University they would have been unemployable within education, for few of them met state requirements for teacher certification.

Differences such as these, however, are merely the obvious ones—and are even perhaps to some extent superficial. What the Nobel prize-winners and their colleagues brought with them was more importantly a new notion of what science and mathematics are, a new notion of what the teacher's task is, and a new notion of what it means to learn, to discover, and to understand.

What was the traditional viewpoint with which these new ideas collided? One outstanding description of the traditional ideology has recently been written by Professor Raymond E. Callahan in his important book *Education and the Cult of Efficiency* (University of Chicago Press, 1962). As Professor Callahan demonstrates clearly, much of the rationale behind the operation of "mass-production" education had been derived—clearly, explicitly, and deliberately—from the "efficiency experts" and "scientific management consultants" (who were *not* scientists), based upon the study of tasks such as loading pig-iron into railroad cars at the lowest possible cost per ton.

Education, growth, maturity, originality, creativity, learning, and understanding are not pig iron. They are not even *similar* to pig iron.

The deepest implications of the "new curriculum" work, I believe, should be sought somewhere among the vibrations and reverberations that continue to emanate from the collision of the point of view of the eminent scientist, and that of the expert on the efficient loading of pig iron.

I have neither the wisdom, the temerity, nor the space to say what all of these implications must be, but it seems safe to say: they will be worth watching for.

Furthermore, they cannot mature and yield a harvest unless they are aided by an intelligent public understanding—which is quite a large enough task to lay before the membership of CBE.

Morris Kline

(Dr. Kline is Chairman of the Department of Mathematics, Washington Square College, New York University. His 1961 article on the "new math" was possibly the first critical one to appear outside the professional journals.)

The content of the new mathematics, as Professor Moise points out in his article, is not new. Though some fringe reform groups have made radical innovations, the leading groups have stuck rather closely to the content of the traditional curriculum. The new feature is the formulation or presentation of the material. This has already been incorporated in several completed high school curricula and is now being carried over to the elementary school curricula. I would agree, too, that the older presentation of arithmetic, algebra and geometry has not been satisfactory. Students have been taught processes or techniques in arithmetic and algebra and the memorization of theorems and proofs in geometry. This mechanical teaching has thrust a meaningless mass of material at the student and as a consequence he has not acquired either knowledge or power.

The central issue is how to present the content of the traditional curriculum, and I and a number of other professors disagree sharply with the innovations popularly called modern mathematics or the new mathematics. I shall try to make clear just what the differences are.

The modern mathematics curricula present mathematics deductively. That is, they start with axioms (as one does in the traditional high school geometry) and then prove theorems. Mathematics in these curricula is a collection of deductive structures. The correct approach, in my opinion, is not deductive but constructive. We should start with simple and concrete situations and with heuristic arguments and only gradually arrive at a deductive organization. Though the nature of this distinction between deductive and constructive will become clearer as we proceed, an example may help. The deductive approach lays down an axiom, called the distributive axiom: that, for any numbers, $a(b + c) = ab + ac$. It follows that $7(5 + 4) = 7 \cdot 5 + 7 \cdot 4$. The constructive approach would seek to convince students through calculation and by using collections of physical objects that $7(5 + 4) = 7 \cdot 5 + 7 \cdot 4$ and that the same holds for any other three numbers, and to encourage the student to make his own generalization. It would then permit him to use this property of numbers freely without being too concerned, at least for quite some time, with the precise logical status of

the principle involved. What is more essential than the particular example is that the students would build up the mathematics and ultimately get to make logical distinctions among definitions, axioms and theorems as they work with mathematics, rather than be forced to adopt a structure imposed from the outside.

The deductive approach rides roughshod over any number of pedagogical difficulties. Thus the operations with negative numbers, a major hurdle for students, are introduced in the deductive approach by insisting that the distributive axiom applies to all numbers. It is then possible to *prove*, for example, that $3 + (-2) = 1$. The constructive approach resorts to the physical meaning of positive and negative numbers to conclude that if we *agree* that $3 + (-2) = 1$, then the operations with negative numbers will be useful in representing physical processes.

More generally, the constructive approach, which employs the genetic approach, tries to build up mathematics somewhat as the subject was developed historically, with full attention to the difficulties which mathematicians themselves experienced in their own development of mathematics. In the historical sequence the deductive organization of any branch of mathematics was the final step, and this step is possible only when the concepts and results are at hand and fully understood. When, therefore, the modern mathematics curricula start the teaching of any branch of mathematics through the deductive approach, they in effect ignore the tens, hundreds and even thousands of years during which concepts and results were slowly acquired, and start where the work really ended. Though one can compress history and avoid many of the wasted efforts and pitfalls, one can not eliminate it.

The modern mathematics curricula are not only deductive; they are *rigorously* deductive. Here too, the history is relevant. After two thousand years of development, mathematicians realized that the ten axioms which Euclid used to construct his geometry are not sufficient, and that some 20 or 25 are needed. But the axioms Euclid omitted were so obvious that he did not realize he was omitting them. (For example, he assumed unconsciously that between any two points of a line there is another point.) And even the greatest mathematician did not realize until about 75 years ago that some axioms were missing. To ask students to recognize as axioms such obvious facts and to use them to prove equally obvious theorems not only complicates the teaching and delays worthwhile topics but

makes students believe that mathematics is concerned with proving the obvious.

The proper approach is to be as intuitive as possible, to adopt axioms which mean something to the student, and to prove only what the student thinks requires proof. The very function of proof is not to prove the obvious in order to meet some sophisticated and purely professional standards and goals, but to prove what is not obvious. Moreover, the rigorous approach is so subtle and artificial that young people are unprepared to take a hand in building it.

The modern curricula present mathematics as an isolated, self-sufficient, pure body of knowledge. By chance, presumably, the deductive structures fit some physical phenomena, and so could be applied to real problems, but even this seemingly fortuitous value is not utilized. Just why anyone wants mathematics or what good it does is not explained. But mathematics is not an isolated, self-sufficient body of knowledge. It exists primarily to help man understand and master the physical world. Mathematics serves ends and purposes. In my opinion, we must constantly motivate mathematics by starting with real problems which cause us to investigate a particular mathematical topic and, having developed the topic, apply the mathematics to show what it can accomplish.

The modern curricula present mathematics as self-generating. For example, negative numbers are introduced to solve an equation such as $x + 6 = 4$. Why anyone should *want* to solve such an equation is never considered. But it is an historical fact that the concepts, techniques, theorems, and even methods of proof were suggested by real situations and real problems. Mathematics grew out of experience in the physical world. This origin of mathematical ideas must be presented. In fact, to delete it is to present mathematics as a series of meaningless marks on paper. The sense of mathematics is physical; without this sense we have nonsense.

The new fashion is to be abstract. The student is asked to learn a general definition, say of a function, and thereafter he is supposed to know just what $y = x$, $y = 2x$, $y = 3x^2$, and so on are all about. The constructive approach recommends that one start with the concrete functions just listed and get the student to work with and understand them. Ultimately he may be able to fashion his own definition of function, but even if he can't do this for some time he will know what functions really are and be able to work with them.

The issue of the abstract versus the concrete may also be clarified

in terms of the theory of sets. Professor Moise brings up the matter of set theory but avoids the real issue. He speaks of the use of the *word* "set" and points out that it means no more than a collection or family of objects. True enough, and in fact no one really objects to the word "set." But the *theory* of sets involves far more than the use of a word. Students are asked to learn operations with sets and the notions of subset, finite and infinite sets, the null set (which is not empty because it contains the empty set), and lots of other notions which are abstract and in fact rather remote from the heart and essence of arithmetic. Yet on this abstract basis students are required to learn arithmetic. The whole theory of sets should be eliminated. On the elementary and high school levels it is a waste of time.

There are other differences between the modern mathematics curricula and what I and others would recommend. The modern curricula insist on precise definitions of practically every word they use and, by actual count, the first two years of the School Mathematics Study Group curriculum ask students to learn about 700 precise definitions. This is pure pedantry. The common understandings which students have acquired through experience are good enough, and formal definitions are usually not needed. After reading the formal definition of a triangle I had to think hard to be sure that it really expressed what I knew a triangle to be. Another point of issue is symbolism. Because some symbolism has made mathematics more effective many naive "mathematicians" now seem to think that the more symbols they introduce the better the mathematics. What they have done is to make a vice out of a virtue.

The essence of the issue I have tried to present above is that the formulators of the modern mathematics curricula believe logic to be the road to understanding whereas intuition, sense impressions, and experience are really the ground on which an understanding of mathematics must be built. Long ago Descartes pointed out that logic is of use only in communicating what we already know, and Pascal said that reason is the slow and tortuous method by which those who do not know the truth discover it.

The proper curriculum has yet to be formulated, and school systems would do well to bide their time. Some efforts in the direction I have been espousing are currently being made, and there is a good chance that the proper reform will soon be available.

Harold M. Bacon

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Professor Moise gives a clear summary of the nature and content of many of the "new mathematics" programs. I believe his most important point is that "while the programs are new and modern, the mathematics contained in them is not. . . . The most important changes have been in the style in which the old content has been formulated and presented." For example, *absolute value* and *inequalities* were not invented within the last ten years; but the rather substantial emphasis now given these topics (even in what many regard as "traditional" or "conventional" programs) is relatively recent. Elementary algebra, plane and solid geometry, trigonometry, and analytic geometry are still essential parts of introductory mathematics—as they have been for a great many years—and "to drop any one of them would be disastrous."[1] Those new programs that have any widespread acceptance include these subjects. A feature of most such programs is a rearrangement of material to reduce what many regard as an artificial compartmentalization of topics; the interplay of algebra and geometry, of two and three dimensional geometry, is exploited to advantage.

Much misunderstanding has arisen, particularly among the lay public, from the labels that have been attached to courses and programs. Thus, "traditional" is used by some writers and speakers as a term of censure which carries the implication of antiquated, inadequate, sterile; "modern," by its very use, implies up-to-date, rich, productive. Attaching a good name to any innovation always helps it to become established as the norm. "Traditional" algebra courses have frequently been condemned for being merely "collections of tricks"—actually a quite unfair general characterization. Too often the "modern" course has emphasized the formalities of structure at the expense of applications and of technique to such an extent that the student emerges with only a halting ability to apply his presumed knowledge. This is not meant to disparage the contemporary effort to bring more understanding to the student of algebra through a much more thorough examination of the logical structure of the

[1] "On the Mathematics Curriculum of the High School," *American Mathematical Monthly*, March, 1962, pp. 189-193 (a statement signed by 65 mathematicians and educators).

subject than was the fashion a generation ago. It is simply to point to the need for caution not to overdo the formalism.

As in any new and exciting movement, there are always protagonists and practitioners who are too zealous, who are swept beyond the borders of reasonable practicality by the enthusiasm of the convert. It is probably good to distinguish between a *number* and a *numeral* (the name of a number); it is quite proper to use precise terminology in defining *function;* it is helpful to introduce such descriptive language as *open sentence* for *equation* and *truth set* for its *roots.* But an overly zealous insistence on elaborate terminology can easily lead to the student's confusing the form for the substance of mathematics, and to creating a false impression that mastery of the language is equivalent to mastery of the concepts and their uses.

Until recently the most widely adopted of the new programs were designed for "college capable" students in secondary schools. Laudable efforts have been, and are now being, made to bring similar programs into a form better adapted to the needs of less gifted students. Revision of programs for the elementary grades is being more and more vigorously pressed. In the not too distant past the junior high school years, especially the seventh and eighth grades, were a mathematical disaster area. Someone characterized seventh grade mathematics as "a review of arithmetic" and eighth grade mathematics as "a review of seventh grade." To many of us, the unconscionable waste of time and the inefficient organization of material in the first eight grades were appalling. One of the chief features of the new programs is an attempt at a reorganization of the curriculum for these grades. Experimentation has been extensive and varied. One innovation that has met with considerable success is the introduction of considerable work in geometry as early as the first and second grades. This has consisted to a large extent of drawing with ruler and compass; students indeed "learn by doing," and the "discovery method" of teaching is exploited to the full; obviously no formal proof in the Euclidean sense is involved. Even work in formal logic has been tried. We hear constant references to the fact that children can learn a great deal more than they have been asked to learn. At the same time, a word of caution is not amiss: the fact that a child *can* learn some particular subject is not, in itself, a justification that he *should* learn it. The body of knowledge available for teaching a child is obviously too large to include

all of it in the curriculum. Choices must be made. In the opinion of many, the early years are the time to develop as fully as possible the child's *intuition* of things mathematical and scientific. More formal treatment can come later.

It is clear that a mathematically literate teacher is essential to the proper handling of the teaching of any program in mathematics —traditional, modern, old or new. In assessing the impact of new programs on the schools, one cannot ignore the statistics: the length of time spent in a teaching career by elementary school teachers is, as a nationwide average, approximately five years. While we are fortunate in having some very well prepared teachers who spend a lifetime in teaching elementary school, the implications of the turnover suggested by these statistics are clear. If we are to have a good program in mathematics, Professor Moise's reference to this problem must be taken very seriously.

In the reorganizing of the mathematics curriculum, both elementary and secondary (to say nothing of collegiate), much effort has been devoted to consolidation of attenuated programs. This is to be commended. For many students, all of the usual material of the conventional twelve years of elementary and secondary school mathematics can be put into the first eleven years—for some students into an even shorter time. But care is needed in doing this. In some programs too many topics are compressed into too short a time; the student must pass on to a new topic before he is really acquainted with the old one. This is unnecessary; in one particular program there was plenty of time for all the topics, but the allocation had provided time for seemingly endless discussion of formal matters, with a rapid introduction in a few pages to such things as permutations, combinations and elementary probability. An eighth-grade student was heard to remark, regarding the continual wrestling with formalism, "I'm certainly sick of this *a over b* stuff," and, regarding the hurried introduction to the other material: "What is all this about? I didn't see what combinations are, and now they're talking about probability." It is a temptation to think that because a subject has once been explained, and a few exercises done, it has become forever afterward the possession of the student. Some of the new programs err in this direction, but experience will soon, we hope, resolve the difficulty.

In the CBE *Bulletin* for December 1961, a number of comments

on Professor Moise's article appear in a letter to the editor from Irving Allen Dodes, Chairman, Department of Mathematics, Bronx High School of Science. In the first paragraph Mr. Dodes says: "The community of mathematicians and mathematics educators is not in full agreement about all points, in particular upon changes in *language*. However, I think all are in substantial agreement with reference to the shifts in emphasis." The first sentence is an understatement, and the second is an overstatement—it depends upon what one means by "shifts in emphasis." What we do agree upon, surely, is the desirability of using the present period of great interest in mathematics to improve to the utmost of our collective ability the mathematics programs in our schools. Reference to the statement "On the Mathematics Curriculum of the High School" should clarify this point.

At a later point in this same letter, Mr. Dodes refers to "the movement away from a convincing but incorrect Euclidean geometry to a more convincing and correct geometry based upon properties of numbers."

One could argue about the interpretation of "incorrect" in this reference to Euclidean geometry; let us hope no one supposes that the conclusions of that geometry are wrong in the sense that conflicting conclusions result from the currently fashionable improvements of the Euclidean axiom system! Whether the system of axioms adopted for the SMSG geometry produces "a more convincing and correct geometry" is a moot point. The answer one gives will depend upon his taste in such matters. This system of axioms ties geometry at its very start to the real number system with its familiar arithmetic. In so doing complications resulting from incommensurable cases are avoided. The measure of area of a given rectangle is postulated to be an unique real number which is the product of the measures of two adjacent sides. This, of course, transfers difficulties about incommensurable lengths to difficulties about the definition of real numbers, in particular of those real numbers that are irrational. So what is gained? To many of us, geometrical (essentially pictorial) intuition is much stronger than intuition about numbers —in fact, to illustrate what a real number is, an appeal is usually made to geometrical intuition. The approach to the concept of area in the Hilbert style is more in the spirit of Euclid—one gets a measure of area of a rectangle by the conventional division into unit squares or rational square subdivisions of the unit square when

the sides of the rectangle are commensurable with the unit length. Although "passage to the limit" in the incommensurable cases is generally regarded as beyond the level of beginners (as is a strictly rigorous definition of a real number), an intuitive treatment that gives a strong feeling for the concept is possible. Many teachers believe that in replacing such a treatment with a reliance upon the beginner's understanding of irrational numbers, the difficulty is merely transferred to something still less appealing to the student's intuition. And let us not suppose that the axiom system used in the SMSG geometry gives a "correct geometry" whereas the axiom system of Hilbert gives an incorrect one! *De gustibus non est disputandum.*

The Advanced Placement Program in mathematics of the College Entrance Examination Board is not often regarded as part of the "new mathematics" program. Nevertheless it is closely involved. Hopefully, new programs will enable strong students to progress more rapidly than might have been the case in past times. If, as is more and more frequently the case, a good student finishes the usual high school mathematics curriculum at the close of the eleventh grade, the question arises: What to do next? One answer is the opportunity to take a college-equivalent course in calculus. The College Entrance Examination Board has a syllabus for such a course, and it offers an examination at the close of the course. Many colleges and universities accept the course and a satisfactory score on this Advanced Placement Examination as equivalent to passing the first year college course in analytic geometry and calculus. It is the opinion of many members of mathematics faculties that this program should be followed by the exceptionally able students who have finished the prerequisite work in those schools which have well-qualified instructors to offer the course, but that calculus is ordinarily not appropriate—at least at present—as a last year or last semester course for the average student who finishes the usual high school program at the end of the eleventh grade. Analytic geometry, and perhaps various other subjects, are generally more suitable. In any case, the nature of the course must necessarily depend upon the capabilities and interests of available teachers.[2] But the fact that increasing numbers of students are well qualified

[2] For a discussion of this subject, see Allendoerfer: "The Case Against Calculus," *The Mathematics Teacher*, November 1963, pp. 482-485.

for such advanced placement courses is an encouraging evidence of growing strength in our school mathematics programs.

General agreement on exactly what should constitute the content and style of school mathematics programs will never be achieved. It is well that it should not be. Dissatisfaction with current practice is what leads to progress and improvement. For a great many years we have had an abundance of the former; it is heartening in our times to observe some, and to hope for more, of the latter.

Suggestions for Further Reading

(The following is a list of some recent articles and pamphlets written for the layman. In addition, there are several recent books on the subject and numerous articles in professional journals.)

Ahrendt, Myrl H.: "How to Switch to New Math," *Nation's Schools*, May 1964, p. 84. A member of the National Science Foundation staff gives nine rules to school administrators.

Burns, Paul C.: "Modern Math: A Brief Guide for Principals," *National Elementary Principal*, April 1964, pp. 55-58. An Associate Professor of Education at the University of Tennessee outlines some of the "new math" content and gives some tips to the elementary school principal on its introduction.

Education Summary, March 1, 1965, pp. 7-8. A short four-paragraph entry reports the criticism of the "new math" made by Dr. Richard W. Hamming, a mathematician with the Bell Telephone Laboratories.

Feinstein, Irwin K.: "Mastering Modern Math," *PTA Magazine*, September 1964, pp. 19-22. A professor of mathematics education at the University of Illinois comments on the content of the "new math" in elementary schools and makes some observations on its adoption. Reprints are available at 15 cents each from PTA Magazine, 700 North Rush St., Chicago, Ill. 60611.

Footlick, Jerrold K.: "The New Math—Philosophy of an Abstract Art," *The National Observer*, March 1, 1965, pp. 12-13. This generally favorable survey will prove very useful to the layman.

Good Housekeeping, January 1965, pp. 132-33. "The New Math Your Children May Be Learning" is a brief, very popular account of the objectives of "new math" and some of the problems of introducing it.

Phillips, Harry L., and Kluttz, Marguerite: "Modern Mathematics and Your Child," U.S. Office of Education, 1963, 28 pp. An entirely favorable account of the "new math." For sale by the Government Printing Office, Washington, D. C. 20402, 20 cents.

"Revolution in School Mathematics," National Council of Teachers of Mathematics, 1962, 96 pp. A report of several regional "orientation" conferences for supervisors and teachers, this booklet gives the rationale for the "new math," a review of course con-

tent, and suggestions for application. For sale by NCTM, 1201 Sixteenth Street, N.W., Washington, D. C. 20036, 50 cents.

Rosskopf, Myron F.: "New Wine in New Bottles—The Demands of a New Mathematics Program," *Independent School Bulletin*, November 1964, pp. 17-22. A professor at Teachers College, Columbia University, analyzes for teachers the "new math" for elementary schools.

Schult, Veryl: "A New Look at the Old Mathematics," *NEA Journal*, April 1964, pp. 12-15. A specialist in mathematics for the U.S. Office of Education describes some aspects of elementary school "new math."

Schwartz, Harry: "New Math Is Replacing Third 'R'," *New York Times*, January 25, 1965, pp. 1, 18. A good survey of the history and present status of the adoption of "new math," with special attention to recent criticisms.

Schwartz, Harry: "'New Math' Leader Sees Peril in Haste,'" *New York Times*, December 31, 1964, pp. 1, 16. A report of the Montreal speech to the National Council of Teachers of Mathematics by Prof. Max Beberman, University of Illinois. (The full text of the speech is unavailable.)

Sharp, Evelyn: "Progress Report on the Mathematics Revolution," *Saturday Review*, March 20, 1965, pp. 62-63, 74-75. An informative but uncritical catalogue of the more sensational successes in experimental work. Almost completely ignores the problems encountered by the typical school.

Strehler, Allen F.: "What's New About the New Math?" *Saturday Review*, March 21, 1964, pp. 68-69, 84. An associate professor of mathematics at Carnegie Tech examines what's new in the secondary school "new math."

Time, January 31, 1964, pp. 34-35. "Inside Numbers" is a general account of the "new math," with some attention to the problems of implementation.

Time, January 22, 1965, p. 38. "The Trials of New Math" reports some of the current difficulties and the criticism being made by Prof. Max Beberman of the University of Illinois, one of the pioneers of "new math."

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